INDIAN STATISTICAL INSTITUTE
End-Semestral Exam
Algebra-II
2017-2018

Total marks: 100
Time: 3 hours

Answer all questions.

(1) State true or false. No marks will be awarded in the absence of proper justification.
(a) Let A be a $n \times n$ matrix such that $A^2 = A$ and $\text{rank}(A) = n$. Then $A = I$.
(b) If row space of a $n \times n$ matrix A equals its column space then $A = A^t$.
(c) Only possible eigenvalues of a 3×3 symmetric orthogonal matrix are 1 and -1.
(d) If A is a complex $n \times n$ matrix such that X^*AX is real for all $X \in \mathbb{C}^n$, then A is Hermitian.
(e) Eigenvalues of a real symmetric matrix are real. [6×5]

(2) Let A, B be $m \times n$ matrices over a field F. Prove that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$. [10]

(3) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A.
(b) Find the minimal polynomial of A.
(c) Is A diagonalizable over $\mathbb{C}$? Give reasons. [6+6+3]

(4) Let V be the space of all real polynomials of degree at most 3.
(a) Prove that

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \ \forall f, g \in V$$

defines a positive definite symmetric bilinear form on V.
(b) Find the orthogonal complement of the subspace of scalar polynomials.
(c) Apply Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$ to find an orthonormal basis of $(V, \langle , \rangle)$. [3+4+8]

(5) (a) Prove that a complex matrix M is normal if and only if there is a unitary matrix P such that P^*MP is diagonal.
(b) Hence show that every conjugacy class in the unitary group $U_n(\mathbb{C}) = \{P \in \mathbb{C}^{n \times n} : P^*P = I_n\}$ contains a diagonal matrix. [12+3]

(6) Let

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}.$$

(a) Find the eigenvalues of A and corresponding eigenvectors.
(b) Find a unitary matrix P such that P^*AP is a diagonal matrix. [5+10]